Effect of Radial Temperature Variation on Axial Mixing in Pipes

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In a pipe reactor axial mixing results in a distribution of residence times of fluid elements in the reactor. Taylor and others have discussed methods of predicting the extent of axial mixing for isothermal flow in pipes; however, in a reactor the heat of reaction and its supply or removal at the wall can cause significant radial variation of temperature and temperature-dependent physical properties. Thus the velocity profile in a reactor may differ from that in a pipe through which fluid flows at constant temperature.

This paper presents an analysis of the effects of radial temperature variations on effective axial diffusion coefficients. In laminar flow it is found that radial temperature variation may increase or decrease the effective diffusivity two- to threefold from that calculated for isothermal flow. At Reynolds numbers greater than 10,000 the diffusivity calculated for constant temperature flow does not differ significantly from that for flow through a reactor tube (with radial temperature variation) at the same Reynolds number, calculated with the viscosity evaluated at the wall temperature. In turbulent flow at Reynolds numbers less than 10,000, the effect of radial temperature variation is important but can be estimated only roughly.

The flow of fluid through a straight pipe is accompanied by mixing in the axial direction. Axial mixing results primarily from variation of the fluid velocity over the pipe cross section and is opposed by radial processes (molecular and eddy). In a pipe reactor axial mixing results in a distribution of residence times of fluid elements flowing through the reactor.

Taylor (*2, 3*), considering both theoretically and experimentally the process of axial dispersion in a fluid flowing through a pipe in either laminar or turbulent isothermal flow, showed that for sufficiently long pipes the dispersion may be successfully treated as apparent axial diffusion about a plane moving through the pipe with the fluid mean velocity. Tichacek, Barkelew, and Baron (*4*) improved upon Taylor's treatment of axial mixing in isothermal turbulent flow, using more exact experimental velocity profiles. They noted the sensitivity of the calculated apparent axial diffusivity to small changes in the fluid velocity profile.

In chemical reactors heat generation (or consumption) by the reaction(s) and cooling (or heating) of the pipe walls create significant radial temperature gradients, which in turn may result in radial variations of physical properties of the flowing fluid sufficient to cause important differences between the velocity profile in the reactor and that anticipated for isothermal flow. If the velocity profile is changed, the apparent axial diffusivity will also be affected. This paper presents the results of an investigation to determine under what flow conditions such radial temperature gradients may have an important effect on the rate of axial mixing and to formulate methods of calculating the magnitude of the effect of a temperature gradient on the apparent axial diffusivity for arbitrary flow and heat-generation conditions.

The investigation has been limited to cases in which the fluid flowing is a liquid. In such cases the primary effect of a radial temperature gradient on axial mixing is through its effect on the viscosity, since other physical properties of the liquid are much less sensitive to temperature change than the viscosity. Effects of temperature on the molecular diffusivity and the fluid density have been neglected in this study.

CALCULATION METHODS

The general method employed has been the calculation of the effective axial diffusivity for a particular flow condition in which the fluid is isothermal and subsequently the calculation of the value of E for a similar flow with radial temperature variation. The ratio of the two values calculated is a quantitative measure of the effect of the imposed temperature gradient on the rate of axial mixing under this flow condition.

The effective axial diffusivity is defined as

$$Q = -\pi a^2 E \frac{\partial C_m}{\partial x} \tag{1}$$

The equation relating the effective diffusivity to the velocity profile and radial diffusivity

$$E = (2/a^2) \int_0^a \frac{1}{r(\epsilon + \mathcal{D})} \left[\int_0^r (u - V) r' dr'\right]^2 dr, \tag{2}$$

has been derived by Tichacek, Barkelew, and Baron (*4*). It is necessary to know u and ϵ as functions of r in order to evaluate E.

Laminar Flow

In pure laminar flow

$$\epsilon(r) \equiv 0 \tag{3}$$

and if the flow is also isothermal,

$$u(r) = 2V(1 - r^2/a^2) \tag{4}$$

With a radial variation in viscosity the radial velocity gradient is related to the wall shear stress and the viscosity:

$$-\frac{du}{dr} = \frac{\tau_w r/a}{\mu(r)} \tag{5}$$

In the calculation of velocity profiles it was assumed that

1. The temperature is either parabolic in radius (corresponding to uniform distribution of heat source or sink due to reaction) or quartic in radius (corresponding to a parabolic distribution of sources or sinks).
2. The relationship between temperature and viscosity is of the form

$$\mu(r) = \frac{K}{B + T(r)} \tag{6}$$

3. The temperature is independent of axial position. (The heat transfer rate at the wall was equal to the heat generation rate in the fluid.) The velocity profile could then be calculated from

$$u = \frac{\tau_w}{K} \int_r^a \frac{r}{a} [B + T(r)] dr \quad (7)$$

The effective axial diffusivity calculated by the use of the velocity profile generated by Equation (7) differed from that calculated for the same flow at constant temperature by a multiplicative factor involving the two constants which characterize the assumed quartic temperature profile.

The assumed viscosity-temperature relationship, Equation (6), was selected because it provided an adequate fit (with ± 0.5 % over an 80°F. range) to handbook data (5) on the viscosity of water. Subsequent comparison with viscosity data for liquids of much higher viscosity showed that Equation (6) was less satisfactory for these liquids; however, calculated effective axial diffusivities were found to be relatively insensitive to changes in radial temperature (and viscosity) profiles. Hence the temperature-viscosity relationship need not fit viscosity data well in order that axial diffusivities may be estimated satisfactorily.

Turbulent Flow

In turbulent flow the velocity gradient is related to the viscosity and momentum eddy diffusivity:

$$-\frac{du}{dr} = \frac{\tau_w r/a}{\mu(r) + \epsilon(r)\rho} \quad (8)$$

Empirical equations for $\epsilon(r)$, given by Deissler (*1*) for fully developed turbulent flow through smooth tubes, were used:

$$\epsilon = n^2 u(a-r)\left[1 - \exp\left(-\frac{n^2 u(a-r)}{\mu_w/\rho}\right)\right] \quad (9)$$

and

$$\epsilon = -N^2 \frac{\left(\dfrac{du}{dr}\right)^3}{\left(\dfrac{d^2u}{dr^2}\right)^2} \quad (10)$$

in which n and N are constants, determined by Deissler as 0.124 and 0.36 respectively.

Equation (9) applies in the region near the wall where viscosity is important in determining the velocity profile:

$$0 < y^+ < 26; \; y^+ = (a-r)\sqrt{\frac{\tau_w \rho}{\mu_w}} \quad (11)$$

Equation (10) is applicable in the central turbulent core, for y^+ greater than 26.

To calculate a velocity profile, Equations (8) and (9) were combined in the form

$$-\frac{du^+}{dz} = \frac{a^+ z}{\mu(r)/\mu_w + (1-z)(n^2u^+a^+)\{1 - \exp[-n^2u^+a^+(1-z)]\}} \quad (12)$$

Equation (11) was numerically integrated from the wall to y^+ of 26 by the use of an assumed value of a^+, and Equation (6) and a parabolic temperature profile were numerically integrated to generate values of $\mu(r)$.

For the region of y^+ greater than 26, Equations (8) and (10) were combined, dropping the effect of viscosity from Equation (8). The resulting equation, in terms of dimensionless variables, may be solved analytically for a velocity profile involving the position z_1, velocity u^+_1, and velocity gradient $(du^+/dz)_1$, at which $y^+ = 26$:

$$u^+ - u^+_1 = 1/N$$

$$\left\{\sqrt{z} - \sqrt{z_1} + A \log\left[\frac{A - \sqrt{z}}{A - \sqrt{z_1}}\right]\right\} \quad (13)$$

$$A = \sqrt{z_1} - 1/2N(du^+/dz)_1 \quad (14)$$

The region at the extreme center of the pipe

$$0 < z < 0.30 = z_2 \quad (15)$$

was fitted by a parabolic velocity distribution. This eliminated subsequent difficulties which would have arisen because the eddy diffusivity corresponding to Equation (13) becomes zero when z is zero. It is justified in this calculation because the contribution of this region to the axial diffusivity is very small. In the region defined by Equation (15) the velocity profile was given by

$$u^+ - u^+_2 = \frac{z_2^2 - z^2}{4Nz_2(A - \sqrt{z_2})} \quad (16)$$

The three sections of the velocity profile, Equations (16), (13), and the integral of Equation (12), were numerically integrated to obtain the mean velocity:

$$V^+ = \int_0^1 2\, u^+ z dz \quad (17)$$

In the calculation of the effective axial diffusivity the assumption was made that the eddy diffusivity for mass transfer was equal to that for momentum transfer. For the three regions of the velocity profile the diffusivities for radial mass transfer were then

$$\frac{\epsilon\rho}{\mu_w} + \frac{1}{N_{Sc}} = \frac{1}{N_{Sc}} + n^2u^+a^+(1-z)$$

$$\{1 - \exp[-n^2u^+a^+(1-z)]\} \quad (18)$$

for $0 < y^+ < 26$

in which N_{Sc} is the Schmidt group $\mu_w/\rho\mathcal{D}$:

$$\frac{\epsilon\rho}{\mu_w} = 2\, N\, a^+z(A - z) \quad (19)$$

for $26 < y^+$ and $z > 0.3$

$$\frac{\epsilon\rho}{\mu_w} = 0.6\, N\, a^+(A - \sqrt{0.3}) \quad (20)$$

for $0 < z < 0.3$

Equation (2) was put into the dimensionless form

$$\frac{E}{DV} = a^+V^+ \int_0^1 \frac{1}{z\alpha}$$

$$\left[\int_0^z (u^+/V^+ - 1)z'dz'\right]^2 dz \quad (21)$$

with

$$\alpha = \frac{\epsilon\rho}{\mu_w} + \frac{1}{N_{Sc}} \text{ for } y^+ < 26 \quad (22)$$

and

$$\alpha = \frac{\epsilon\rho}{\mu_w} \text{ for } y^+ > 26 \quad (23)$$

The indicated integrations were carried out numerically to obtain a value of E/DV for the value of a^+ and the temperature profile used in the calculation.

Six values of E/DV were calculated for isothermal flow with a^+ equal to 130, 260, and 650, each with Schmidt numbers of 100 and 1,000.

Deissler's values for n and N apply strictly only for fully developed turbulent flow. For the transition region some adjustment is necessary to make Equation (12) fit the known velocity profiles (*4*). n = 0.0933 and N = 0.279 were found to be suitable for a^+ = 130 and were used in all calculations for this case. Deissler's values of 0.124 and 0.36 were retained in calculations at higher Reynolds numbers.

At a^+ of 130 and 650 E/DV was calculated for flow with parabolic temperature profiles such that the ratio of the viscosity at the center line to that at the wall was 0.1 and 10.0. Each calculation was made at Schmidt numbers of 100 and 1,000.

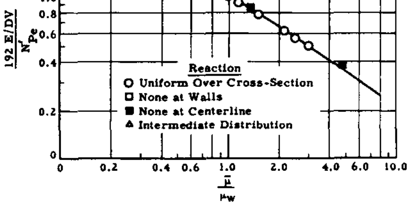

Fig. 1. Effective axial diffusivities for laminar flow of liquids in smooth tubes.

RESULTS

Laminar Flow

For isothermal laminar flow the result given by Taylor (2) may be put into the form

$$\frac{E}{DV} = \frac{DV/\mathfrak{D}}{192} = \frac{N'_{Pe}}{192} \quad (24)$$

For nonisothermal flow the ratio $(E/DV)_{\text{nonisothermal}}/(E/DV)_{\text{isothermal}}$ is conveniently expressed as a function of some characteristic of the radial viscosity profile. In Figure 1 this ratio is plotted vs. $\bar{\mu}/\mu_w$, the ratio of the viscosity of the fluid at the mixed mean temperature to the viscosity at the wall temperature. The effective axial diffusivity is relatively insensitive to the actual temperature profile. Calculated values for the extreme quartic temperature profiles (no reaction at the walls or no reaction at the center) lie within ± 10% of the curve in Figure 1 drawn through values calculated for parabolic temperature distributions (uniform heat generation or consumption over the pipe cross section).

Figure 1 shows that as the ratio $\bar{\mu}/\mu_w$ decreases, corresponding to extreme heat generation by reaction in the fluid and a high cooling rate at the walls, the effective axial diffusivity increases toward a limiting maximum approximately 2.6 times its value for comparable isothermal flow (at the same Peclet number for molecular diffusion). At the other extreme, with heat consumption in the liquid and heating at the pipe wall, E decreases toward zero as a limit.

The most important limitation on the applicability of Figure 1 is in the regions of very high or very low values of $\bar{\mu}/\mu_w$, where it is anticipated that the viscosity-temperature relationship, Equation (6), is no longer appropriate. If a value of E/DV is required for such conditions, it is recommended that it be determined by integration from Equations (2) and (5) by means of a parabolic temperature profile and an empirical viscosity-temperature relationship appropriate to the system being considered.

Turbulent Flow

Effective axial diffusivities were calculated at two values of the friction velocity Reynolds number, $a^+ = 650$ and 130. For each value E/DV was calculated for N_{Sc} of 100 and 1,000 with ratios of viscosity at the wall to viscosity at the center line of 0.1, 1.0, and 10.0. Figure 2, for $N_{Sc} = 1{,}000$, compares the values of E/DV calculated for nonisothermal flow with curves based on values of E/DV calculated by Tichacek, Barkelew, and Baron (4) for isothermal turbulent flow. The abscissa is the usual pipe-flow Reynolds number $DV\rho/\mu$, with the viscosity evaluated at the wall temperature.

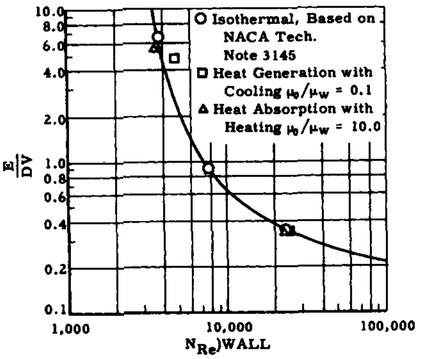

Fig. 2. Effective axial diffusivities for turbulent flow of liquids in smooth tubes at Schmidt number of 1,000.

For Reynolds numbers greater than 10,000 E/DV is not significantly affected by radial temperature gradients. The nonisothermal points in Figure 2 at N_{Re} of 20,000 to 25,000 deviate from the isothermal curve by only 3 to 6% in E/DV, much less than the 20 to 30% uncertainty associated with the calculation of E/DV for isothermal flow. Similar results were obtained for N_{Sc} of 100. Therefore Tichacek, Barkelew, and Baron's values of E/DV are recommended for turbulent flow at Reynolds numbers greater than 10,000, even though the heat generated by reaction causes significant radial variations in viscosity. In the selection of the value of E/DV the Reynolds number with the viscosity evaluated at the wall temperature should be used.

Transition Region

In flow with a radial temperature gradient it is not possible to characterize the transition between laminar and turbulent flow by a narrow range of Reynolds numbers. Temperature distributions with minimum viscosity at the wall stabilize laminar flow, and those with maximum viscosity at the wall tend to decrease the stability of laminar flow.

The points in Figure 2 at Reynolds numbers between 3,500 and 5,000 lie in a region in which either laminar flow may prevail or else the effect of the laminar sublayer region of the pipe is important in determining E/DV and thus the effect of the temperature gradient on the level of turbulence in the flow cannot be estimated accurately. These values of E/DV were calculated on the assumption that the flow is turbulent and that Equations (9) and (10) are applicable with the same values of the constants n and N that were used in calculating E/DV for isothermal flow. For the case in which heat is generated in the liquid and the walls are cooled, the calculated E/DV is larger than the true E/DV. In the alternate case the calculated E/DV is probably less than the true value.

NOTATION

A	=	constant characteristic of a turbulent velocity profile
a	=	pipe radius
a^+	=	dimensionless pipe radius $a\sqrt{\tau_w\rho}/\mu_o$
B	=	a constant
C_m	=	area mean concentration
D	=	pipe diameter
$\mathfrak{D}$	=	molecular diffusivity
E	=	effective axial diffusivity
K	=	a constant
n	=	a constant
N	=	a constant
N'_{Pe}	=	Peclet number, $DV/\mathfrak{D}$
N_{Re}	=	Reynolds number, $DV\rho/\mu$
N_{Sc}	=	Schmidt group, $\mu/\rho\mathfrak{D}$
Q	=	net transport of solute past reference plane moving with velocity V
r	=	radial coordinate
u	=	time-average velocity at a point
u^+	=	dimensionless local velocity, $u/\sqrt{\tau_w/\rho}$
V	=	mean velocity over pipe cross section
V^+	=	dimensionless mean velocity $V/\sqrt{\tau_w/\rho}$
x	=	axial distance from a reference plane moving with velocity V
y^+	=	dimensionless distance from wall $(a-r)\sqrt{\tau_w\rho}/\mu_w$
z	=	dimensionless radial coordinate, r/a

Greek Letters

α	=	total radial diffusivity
ϵ	=	coefficient of eddy transport
ρ	=	density
τ_w	=	wall shear stress
μ	=	local viscosity
$\bar{\mu}$	=	viscosity at mean fluid temperature
μ_o	=	viscosity at temperature at pipe center line
μ_w	=	viscosity of fluid at the wall temperature

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